Optimality Of Control

Part of a set of lecture notes on Introduction to Robust Control by Ming T. Tham (2002)

Introduction

From our discussion on sensitivity functions, we know that perfect control can never be achieved in practice. We have also discussed the fact that compromises are often made: between performance and stability; between excessive sensitivity to noise and attainment of control objectives; between good set-point tracking and good disturbance rejection. Because of all the compromises that may have to be made, in assessing the performance of any particular scheme, we therefore have to answer the question as to how 'good' the implemented control is.

How Good is Good!

What is 'good' control? With level systems for example, 'good' may mean the ability to maintain a level quickly. Offsets can be tolerated as long as the level of the system is maintained on average. In this case, proportional controllers are often used. With reaction systems, temperatures may have to be maintained exactly, and thus offset free control schemes have to be adopted. Additionally, overshoots may not be tolerated on some temperature loops because the product may be thermally degraded. On the other hand, we may require changes in temperature demands be achieved quickly, but fast response times are usually accompanied by overshoots.

'Good control' is therefore quite a subjective judgement. We always have to consider:

1) the characteristics of the process being controlled,
2) the constraints imposed by operating requirements and of course,
3) the cost of achieving desired control objectives.

If the desired control performance can be achieved under all these constraints, then we say that the system is under 'optimal control'. Thus, when we attempt to assess the optimality of a particular control scheme, we would normally judge its performance against some user defined objective.

The user defined objective is normally a mathematical expression and is called the *cost function* or *objective function*. It may include descriptions about the constraints that the control system has to abide by. Typical constraints include:

1) Limits on allowable minimum and maximum values for manipulated inputs and controlled outputs ('hard' constraints);
2) limits on the maximum rate of change for inputs ('rate' or 'soft' constraints);
3) limits on outputs that are not directly controlled ('associated' constraints).

The controller and its parameters are determined by solving of this optimisation problem.

Scope of Discussion

In this course, we shall look at two optimal control strategies:

1) Linear Quadratic Optimal Control
2) H-∞ Optimal Control

For simplicity, we shall restrict our discussion to unconstrained optimal control problems.

Linear Quadratic Optimal Control

Linear Quadratic Optimal Control (or simply LQ control) strategies have been around for many years. Its objective is to find a control that will minimise the following integral:

$$J = \int_0^\infty e^2(t)dt$$

where $e(t) = r(t) - y(t)$; $y(t)$ is the controlled output and $r(t)$ is the set-point. The LQ control objective function is therefore commonly called the Integral of Squared Error (ISE) criterion. Rather than minimising a point error, the aim is to minimise the squared set-point tracking error over a period of time. Thus, the LQ strategy may be regarded as trying to find the controller that will minimise the average magnitude of this error.

Suppose the error sequence $\{e(t), t = 0,1,2,\ldots,\infty\}$ is stored in a vector **e**. Then, as the integration essentially sums the squared errors from time zero to time infinity, the cost function J can be represented more compactly as:

$$J = \mathbf{e}^{\mathrm{T}}\mathbf{e}$$

Now, since the 2-norm of **e** is

$$\|\mathbf{e}\|_2 = \sqrt{\mathbf{e}^{\mathrm{T}}\mathbf{e}} \qquad \text{(see notes on Vector Norms)}$$

then $\quad J = \|\mathbf{e}\|_2^2$

Thus the design of the LQ control strategy can be expressed mathematically as $\min_{G_c} J$ or more specifically:

$$\min_{G_c} \|\mathbf{e}\|_2^2$$

which reads "find a G_c that will minimise $\|\mathbf{e}\|_2^2$".

As a result of the use of the 2-norm, LQ optimal control is nowadays also commonly referred to as H_2 control.

From our discussion on the sensitivity function, we know that

$$e(t) = \varepsilon(t)[r(t) - d(t)]$$

where $\varepsilon(t)$ is the sensitivity function in the time-domain, and $d(t)$ is a disturbance input. The LQ control problem may therefore also be posed in terms of the sensitivity function. This results in:

$$\min_{G_c} \|\varepsilon q\|_2^2$$

Here, q, is a weighting which is chosen depending on whether the input to the system is a set-point or a disturbance. That is, the user chooses a value for q so as to tailor the behaviour of the LQ controller in following set-point changes or rejecting disturbances. Therefore, q, can be interpreted as a tuning parameter that can be used to reduce overshoots, increase the speed of response, etc.; much like the parameters of the familiar three term PID controller.

If the LQ problem is posed in the frequency domain by use of Parseval's Theorem, we obtain:

$$\min_{G_c} \frac{1}{2\pi} \int_{-\infty}^{\infty} |\varepsilon(j\omega)q(j\omega)|^2 d\omega$$

The significance is that only the magnitude of q is important, and not its phase properties.

H-∞ Optimal Control

Compared to the LQ control scheme, H-∞ control is a more recent development. Its objective is to minimise the worst error that can result from any input and is formulated as:

$$\min_{G_c} \|\mathbf{e}\|_\infty \qquad (\|.\|_\infty \text{ is the } \infty\text{-norm})$$

which reads "find a controller G_c that minimises the worst (maximum) error [in the error sequence contained in the vector **e**]". In this sense, H-∞ control will provide tighter control than that provided by applying H_2 or LQ control.

As with LQ control, H-∞ control can be re-formulated in terms of the sensitivity function as:

$$\min_{G_c} \|\boldsymbol{\varepsilon} q\|_\infty$$

In the frequency domain, this objective is written as:

$$\min_{G_c} \sup_{\omega} |\varepsilon(j\omega)q(j\omega)|$$

(‘sup’ is the short form for ‘supremum’ and is the mathematical description for an ‘upper bound’)

Again, only the magnitude of q matters, and can selected by the user to provide the desired response characteristics, depending on whether the input is a set-point or a disturbance. However, we can make use of q in another way when we are applying H-∞ control.

Suppose the ‘optimal’ value of the H-∞ objective function is k, i.e.

$$\|\boldsymbol{\varepsilon} q\|_\infty = k$$

or $$\|\boldsymbol{\varepsilon} q\|_\infty \equiv \sup_{\omega} |\varepsilon(j\omega)q(j\omega)| = k$$

then $|\varepsilon(j\omega)| \le k|q(j\omega)|^{-1}$ for all values of ω.

k is a scalar. Therefore, it can be ‘absorbed’ into the magnitude term on the left hand side of the inequality to give

$$|\varepsilon(j\omega)| < |q(j\omega)|^{-1}$$

which is equivalent to

$$\|\boldsymbol{\varepsilon} q\|_\infty < 1$$

Also recall that for perfect control, the sensitivity function, ε, is zero. Therefore, the above result has practical significance because the designer can now use the weighting q to impose an upper limit on the sensitivity function. It is no longer merely a weighting that is used to shape the response characteristics of the controlled output. For this reason, the H-∞ control objective is often expressed as:

$$\min_{G_c} \|\boldsymbol{\varepsilon} q\|_\infty < 1$$

which reads "find a controller G_c such that $\|\boldsymbol{\varepsilon} q\|_\infty < 1$" or

$$\min_{G_c} |\varepsilon(j\omega)| < |q(j\omega)|^{-1} \text{ for all values of } \omega$$

which reads "find a controller G_c such that $|\varepsilon(j\omega)|$ is less than $|q(j\omega)|^{-1}$ at all frequencies".

The *q* weighting in H-∞ control

Expressing the control objective as a bound on the sensitivity function has much practical appeal. For example,

a) the designer may wish to specify a minimum bandwidth (the frequency range of interest). The bandwidth is usually defined as the frequency ω_b at which $|\varepsilon|$ exceeds $\frac{1}{\sqrt{2}}$ (≈-3dB). Clearly q can be chosen such that the desired bandwidth would be achieved, i.e.

$$|\varepsilon(j\omega)| < \frac{1}{\sqrt{2}} \text{ for all } \omega < \omega_b.$$

b) the designer may wish to limit the peak value of the sensitivity function so as to suppress excessive amplification of disturbance effects. Again the q weighting can be chosen to achieve this explicitly. For the conventional feedback control loop, recall that $|G_c(j\omega)G_p(j\omega)|$ is small at high values of ω. This implies

$$|\varepsilon(j\omega)| = \left|\frac{1}{1+G_c(j\omega)G_p(j\omega)}\right| \approx 1 \text{ for large } \omega$$

This means that tight performance specifications ($|\varepsilon(j\omega)| \approx 0$) are only meaningful at low frequencies where $|G_c(j\omega)G_p(j\omega)|$ is large, in which case

$$|\varepsilon(j\omega)| \approx |G_c(j\omega)G_p(j\omega)|^{-1}$$

Since the H-∞ performance specification is given by

$$|\varepsilon(j\omega)| < |q(j\omega)|^{-1} \text{ for all values of } \omega$$

then

$$|G_c(j\omega)G_p(j\omega)| > |q(j\omega)| \text{ for low values of } \omega.$$

In other words, the loop gain has to be made larger than $|q(j\omega)|$.

Summary

- The optimality of a controller does not necessarily mean that it is useful in practice.
- The usefulness of an optimal control scheme depends on how well the controller design has taken into account realism in the mathematical formulation of the problem.
- The LQ control scheme attempts to minimise the average magnitude of the error between controlled output and set-point.
- The H-∞ controller tries to minimise the worst error that can occur.
- The H-∞ controller can provide for tighter control than the LQ controller
- The formulation of the H-∞ control strategy allows the control systems designer to explicitly accommodate the compromises that have to be made in the implementation of the scheme to real processes. Therefore, the H-∞ control methodology offers a better

framework for the design and analysis of robust control strategies than that provided by the LQ scheme.