Brief Notes On Vector Norms

Part of a set of lecture notes on Introduction to Robust Control by Ming T. Tham (2002)

- A vector norm is a 'metric' or measure of distance in a vector space. It is denoted by the symbol $\|.\|_p$.

The subscript '*p*' denotes how the norm has been defined.

If the subscript '*p*' appears explicitly, then the norm is referred to as the 'p-norm'.

- For any n-dimensional vector $\mathbf{x} = \left[x_1, x_{2,} x_3, \cdots, x_n\right]^T$, any norm of **x** must satisfy the following properties:

  1) $\|\mathbf{x}\|_p \geq 0$

  2) $\|\mathbf{x}\|_p = 0 \quad iff \ \mathbf{x} = \mathbf{0}$, i.e. **x** is the 'null' vector with $x_i = 0, \quad i = 1,2,3,\ldots,n$

  3) For any scalar α, $\|\alpha\mathbf{x}\|_p = |\alpha|.\|\mathbf{x}\|_p$

  4) If **y** is another n-dimensional vector, then $\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$. This relationship is known as the 'triangle inequality'.

- In control systems analysis, the two most commonly used norms are the

  2-norm ($\|.\|_2$) also called the Euclidean Norm

and the

  ∞-norm ($\|.\|_\infty$)

- The 2-norm of a vector $\mathbf{x}$ is given by $\|\mathbf{x}\|_2 = \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{1/2}$ while the ∞-norm is given by $\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$

- The 2-norm and the ∞-norm of an n-dimensional vector $\mathbf{x}$ satisfy the following inequalities:

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \sqrt{n}\|\mathbf{x}\|_\infty$$

Examples:

Let $\mathbf{x} = [-1, 1, -2]^T$. Then,

$$\|\mathbf{x}\|_2 = \sqrt{(-1)^2 + 1^2 + (-1)^2} = \sqrt{6}$$

while

$$\|\mathbf{x}\|_\infty = \max\{|-1|, |1|, |-2|\} = 2$$

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