Differential Equations

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Introduction

In order to understand and effectively design chemical processes as well as their associated control systems they must be understood. Mathematical modelling involves the representation of physical devices such as tanks, reactors, columns, etc as well as the control hardware. Developing mathematical models of chemical processes normally involves the use of differential equations. Once written, these equations must be solved to indicate the performance of a particular system.

The aim of these notes is to revise differential equations and their solution. Attention is restricted to the investigation of the behaviour of systems that may be described using 1st order ordinary differential equations. At the end of this section of the course you should understand the behaviour of 1st order DE's when subject to common process input changes.

What is a differential equation?

Differential equations (DE's) are equations in the usual sense however at least one of the terms will include a rate of change (differential) term. The following equations are examples of DE's,

$$\frac{dy}{dt} = 3 \quad \text{and} \quad \frac{d^2y}{dt^2} + \frac{dy}{dt} + 6 = 3 \tag{1}$$

where 'y' could represent a process output such as temperature, flow or level and 't' could represent time. A DE provides information regarding the rate of change of a process variable with respect to time. This can provide important information regarding the current and future operation of a process. For a level system this may allow the prediction of liquid overflow from a vessel and for a temperature system prediction of excessive temperature fluctuations, for instance.

The term 'order of a DE ' refers to the highest differential term that is involved. So the two equations are 1st and 2^{nd} order. To understand how 'y' changes with respect to 't', the DE must be analysed. One method that may be used to analyse DE's is direct solution.

Direct solution of DE's

The basic principles behind the solution of a DE are common and can be introduced by consideration of a simple example,

$$\frac{dy}{dt} = 3 \qquad (2)$$

Through integration, a solution to this DE is,

$$y = 3t + c \qquad (3)$$

Equation (3) is called the general (or complementary) solution of the DE. It represents a 'family' of lines all with gradient '3'. To obtain a specific (or particular) solution to the problem we require boundary (or initial) conditions. Suppose that the initial conditions are t = 0, y = 2, then by substitution of the boundary conditions into Equation (3), the constant is calculated as c = 2. Therefore the particular solution of this DE is given by,

$$y = 3t + 2 \qquad (4)$$

Direct solution of 1^{st} order DE's

There are various types of DE (the reader should refer to mathematics texts for further information). These notes concentrate on the solution of 1^{st} order DE's that are subjected to a 'forcing' input function. The particular structure is,

$$\frac{dy}{dt} + Ay = B \qquad (5)$$

where A and B are constants or functions of 't' only.

To solve this type of DE an integrating factor is used.

The integrating factor

Let 'I' be a function of 't'. Multiply equation (5) by I,

$$I\frac{d(y)}{dt} + IAy = IB \qquad (6)$$

Now as, $\frac{d(Iy)}{dt} = I\frac{dy}{dt} + y\frac{dI}{dt}$ [1], equation (6) may be written as,

$$\frac{d(Iy)}{dt} - y\frac{dI}{dt} + IAy = IB \qquad \text{or} \qquad \frac{d(Iy)}{dt} + y(IA - \frac{dI}{dt}) = IB \qquad (7)$$

The integrating factor ‘I’ is chosen so that,

$$\frac{dI}{dt} = IA \qquad (8)$$

In other words, I is given by[2],

$$I = \exp(\int Adt) \qquad (9)$$

Substituting for equation (8) followed by equation (9) in equation (7) gives,

$$\frac{d}{dt}\exp(\int Adt)y = B\exp(\int Adt) \qquad (10)$$

which can be integrated (believe it or not!).

Worked Example 1

1. Solve the DE, $\frac{dy}{dt} + 3y = 1$ with boundary conditions, t = 0, y = 0.

Solution: The integrating factor is,

$$I = \exp(\int 3dt) = \exp(3t)$$

So the DE may be written as,

$$\frac{d(e^{3t}y)}{dt} = e^{3t}$$

Integrating with respect to ‘t’ gives,

$$e^{3t}y = \frac{1}{3}e^{3t} + c \qquad \text{or} \qquad y = \frac{1}{3} + ce^{-3t}$$

Substituting for the initial conditions, t = 0, y = 0 gives c = -1/3 so the solution to the DE is given by,

$$y = \frac{1}{3}(1 - e^{-3t})$$

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[1] This is the product rule for differentiation

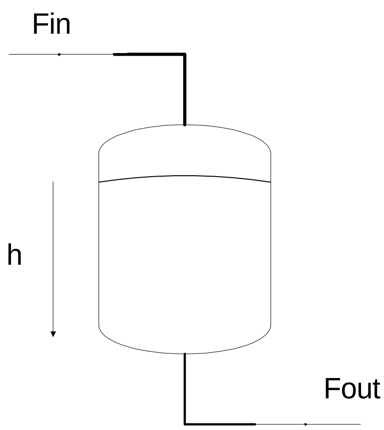
[2] $\frac{1}{I}\frac{dI}{dt} = A \Rightarrow \ln(I) = \int Adt$

Exercise 1

Using Matlab, plot the trajectory of $y = \frac{1}{3}(1 - e^{-3t})$ with t = 0, y = 0.

Exercise 2

Consider the following tank system,



Liquid tank system

The flow into the vessel is given by Fin (m^3/min), the flow out of the vessel, Fout (m^3/min), the cross-sectional area of the tank is A (m^2) and the height of liquid in the tank is h (m). The differential equation describing the change in liquid height with respect to time is given by,

$$A\frac{dh}{dt} = F_{in} - F_{out}$$

The flow out of the vessel is related to the height of liquid in the vessel. The relationship is known to be,

$$F_{out} = \alpha\sqrt{h}$$

where α is a constant. However, it will be assumed that the relationship is given by,

$$F_{out} = \alpha h$$

i.e. a linear relationship. Therefore the differential equation that describes the change in level within the tank is given by,

$$A\frac{dh}{dt} = F_{in} - \alpha h$$

The tank dimensions are A= 1 m^2, total height = 1m. The steady-state liquid inlet flow is 0.2m^3/min with a nominal level of 0.5m and the constant α = 0.2.

The tank has been equipped with instrumentation so that if the level exceeds a value of 0.95m or falls below a value of 0.05m an alarm will sound to alert process operators.

(1) Analyse the level system and show the effect that a step change in inlet flow of 0.05 m^3/min has on the level and determine whether an alarm would sound.
(2) Plot the level of liquid in the tank versus time to verify the results of the analysis.

Writing a 1st order DE in a process specific form

The time dependant behaviour of chemical process systems can often be represented by a 1st order differential equation. Normally, this differential equation is written in the following form,

$$\tau \frac{dy}{dt} + y = k_p u \qquad (11)$$

where 'τ' is known as the system time constant. This is a fundamental time property of the system and K_p is known as the process gain and represents the 'steady-state' relationship between the input 'u' and the output 'y', i.e $y = k_p u$. (these terms will be defined in more detail later in the notes).

Solving the DE if u = constant (a step change)

Following the procedure outlined above the solution to equation (11) is calculated as follows. The integrating factor is,

$$I = \exp(\int \frac{1}{\tau} dt) = \exp(\frac{t}{\tau}) \qquad (12)$$

So the DE may be written as,

$$\frac{d(e^{\frac{t}{\tau}} y)}{dt} = e^{\frac{t}{\tau}} \frac{k_p}{\tau} u$$

Integrating with respect to 't' gives,

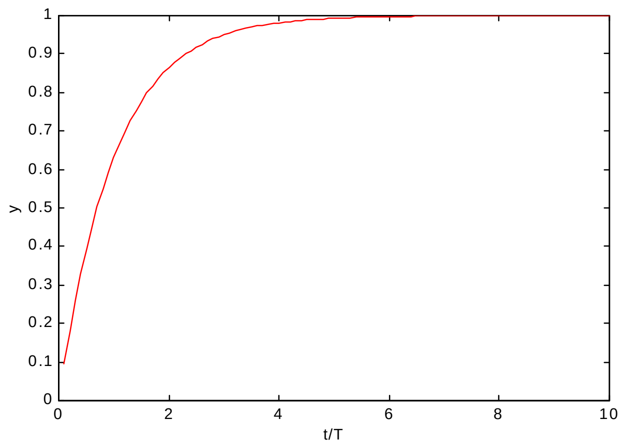
$$e^{\frac{t}{\tau}} y = \tau e^{\frac{t}{\tau}} \frac{k_p}{\tau} u + c$$

$$y = k_p u + e^{\frac{-t}{\tau}} c$$

Suppose that the initial conditions are t = 0, y = 0, then c = -k_pu and the solution to the differential equation is given by,

$$y = k_p u(1 - e^{\frac{-t}{\tau}}) \tag{13}$$

Equation (13) describes the response of the first order DE when subject to a step change in the input (initially the input was zero and at time zero it is changed to a new value, u). Plotting equation (13) reveals the following behaviour,



Response of a 1st order system to a step change in input (u =1, kp = 1, τ = 10)

The output gradually increases to a new value. Characteristics of this trajectory can be predicted by analysis of equation (13).

<u>The new 'steady-state' value of y.</u>

Initially, y = 0. After a change in the input (u= 1 at t = 0) the output will change and it would be useful to calculate the new value of 'y' when the system reaches its new 'steady-state' conditions (e.g. this could allow the prediction of future process conditions). To do this, assume t = ∝ and from equation (13),

$$y = k_p u$$

Therefore the new steady-state value of y is calculated using the input multiplied by the process gain. In the example the gain k_p = 1 and u = 1 therefore the final value of y is also 1.

The time constant, τ

This is a function of the physical properties of a system. It is the factor that determines the speed of response of the system, i.e. how long it takes to reach the new steady-state. Setting t = τ, in equation (13) gives,

$$y = 0.632k_p u$$

In other words, the time constant is defined as the time it takes for the system to move 63.2% of the way from the initial to the new steady-state value (refer to the figure above to verify this fact).

Solving the DE if u = αt (a ramp change where α is a constant and t is time)

The integrating factor is,

$$I = \exp(\int \frac{1}{\tau} dt) = \exp(\frac{t}{\tau}) \tag{14}$$

So the DE, equation (11), may be written as,

$$\frac{d(e^{\frac{t}{\tau}} y)}{dt} = e^{\frac{t}{\tau}} \frac{k_p}{\tau} \alpha t$$

Integrating with respect to ‘t’ gives (see Appendix A for details),

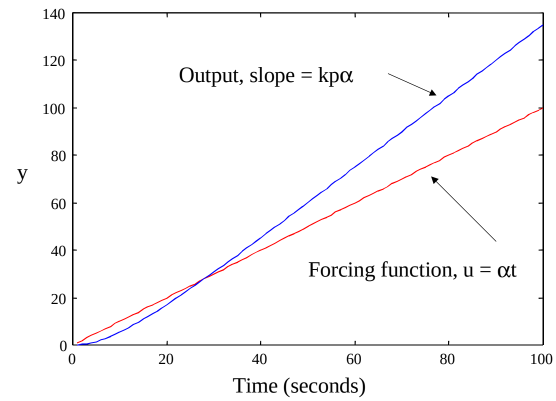
$$e^{\frac{t}{\tau}} y = k_p \alpha e^{\frac{t}{\tau}} (t - \tau) + c$$

$$y = k_p \alpha (t - \tau) + e^{\frac{-t}{\tau}} c \tag{15}$$

If the initial conditions are t = 0, y = 0, then c = $k_p\alpha\tau$ and the solution to the differential equation is given by,

$$y = k_p \alpha (t - \tau + \tau e^{\frac{-t}{\tau}}) = k_p \alpha \tau (\frac{t}{\tau} - 1 + e^{\frac{-t}{\tau}}) \tag{16}$$

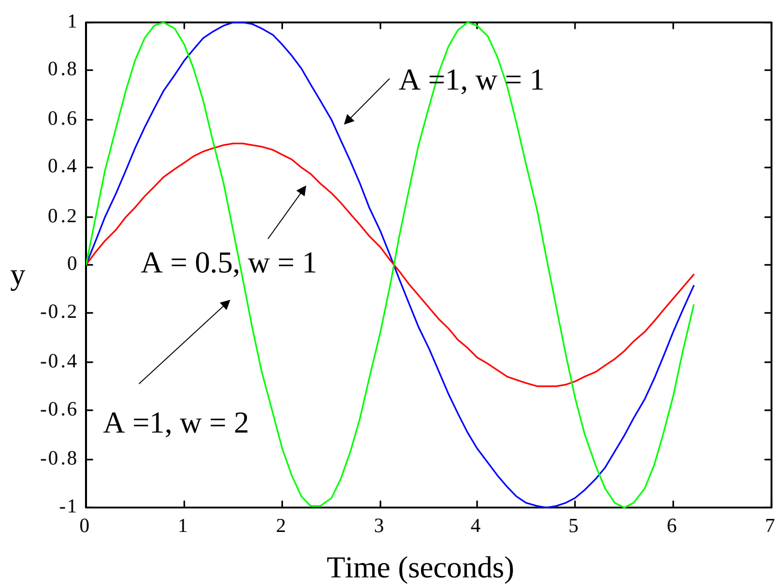
Plotting equation (16) reveals the following behaviour,



Response of a 1st order system to a ramp change in input
(kp = 1.5, α = 1, τ = 10)

Solving the DE if u = Asin(wt) (a sinusoidal input, amplitude A, frequency w)

The following figure demonstrates the characteristics of a sinusoidal input,



Response of u = Asin(wt)

'A' represents the 'amplitude' of the input signal. Two of the signals have amplitude of 1 while the other has amplitude of 0.5. 'w' represents the 'frequency' of the sine wave. The three signals have frequencies equal to 0.5, 1 and 2.

Appendix A details the solution of the DE which is given by,

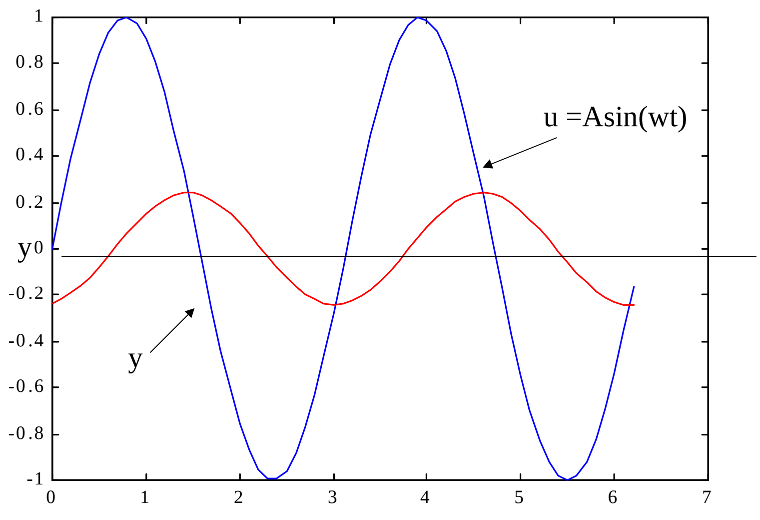
$$y = \frac{k_p A}{1 + w^2\tau^2}\left[\sqrt{1 + w^2\tau^2}\sin(wt + \phi) + w\tau e^{\frac{-t}{\tau}}\right] \quad (17)$$

where $\phi = \tan^{-1}(-w\tau)$

When $t \to \infty$, the solution simplifies to,

$$y = \frac{k_p A}{\sqrt{1 + w^2\tau^2}}\sin(wt + \phi) \quad (18)$$

The output behaviour of a first order system when subjected to a sinusoidal input signal is plotted below,



Response of y to a sinusoidal input u = Asin(wt)
(A = 1, w = 2, kp = 1, τ = 2)

Amplitude ratio and phase lag

The amplitude of the input signal is A, while the amplitude of the output signal is given by

$$y = \frac{k_p A}{\sqrt{1 + w^2\tau^2}} \quad (19)$$

Their ratio is therefore,

$$y = \frac{k_p}{\sqrt{1 + w^2\tau^2}} \quad (20)$$

In other words, the amplitude ratio is a function of the process parameters (k_p and τ) as well as the frequency of the input signal, w. For a 1st order differential equation this amplitude ratio is never greater than 1.

Looking at the figure it is obvious that the output lags behind the input signal. This characteristic is known as phase lag and is given by $\phi = \tan^{-1}(-w\tau)$. For the example given ϕ = 1.3 seconds.

Summary

These notes provide an introduction to DE's. The focus has been 1st order DE's as the dynamic (time dependant) characteristics of many chemical processes can be approximated using this model. The response of a 1st order DE to three standard inputs has been shown in detail.

Appendix A

a) Solution of $\frac{d(e^{\frac{t}{\tau}}y)}{dt} = e^{\frac{t}{\tau}}\frac{k_p}{\tau}\alpha t$

The right hand side (RHS) of the equation must be integrated by parts[3],

Letting, $\frac{dv}{dt} = e^{\frac{t}{\tau}} \Rightarrow v = \tau e^{\frac{t}{\tau}}$ and $u = \left(\frac{k_p\alpha}{\tau}\right)t \Rightarrow \frac{du}{dt} = \left(\frac{k_p\alpha}{\tau}\right)$ then,

$$e^{\frac{t}{\tau}}y = \int e^{\frac{t}{\tau}}\frac{k_p}{\tau}\alpha t dt = \left(\frac{k_p\alpha}{\tau}\right)t\tau e^{\frac{t}{\tau}} - \int \tau e^{\frac{t}{\tau}}\left(\frac{k_p\alpha}{\tau}\right)dt + c = \left(\frac{k_p\alpha}{\tau}\right)t\tau e^{\frac{t}{\tau}} - k_p\alpha\tau e^{\frac{t}{\tau}} + c$$

Re-arranging gives,

$$y = k_p\alpha(t-\tau) + ce^{\frac{-t}{\tau}}$$

Applying the boundary conditions that at t = 0, y = 0 gives $c = k_p\alpha\tau$ therefore,

$$y = k_p\alpha(t - \tau + \tau e^{\frac{-t}{\tau}}) = k_p\alpha\tau(\frac{t}{\tau} - 1 + e^{\frac{-t}{\tau}})$$

b) Solution of $\dfrac{d(e^{\frac{t}{\tau}}y)}{dt} = e^{\frac{t}{\tau}}\dfrac{k_p}{\tau}A\sin(wt)$

As before, the RHS of the equation must be integrated by parts,

Letting, $\dfrac{dv}{dt} = e^{\frac{t}{\tau}} \Rightarrow v = \tau e^{\frac{t}{\tau}}$ and $u = \left(\dfrac{k_p A}{\tau}\right)\sin(wt) \Rightarrow \dfrac{du}{dt} = \left(\dfrac{k_p A}{\tau}\right)w\cos(wt)$ then,

$$e^{\frac{t}{\tau}}y = \int e^{\frac{t}{\tau}}\frac{k_p}{\tau}A\sin(wt)dt = \left(\frac{k_p A}{\tau}\right)\sin(wt)\tau e^{\frac{t}{\tau}} - \int \tau e^{\frac{t}{\tau}}\left(\frac{k_p A}{\tau}\right)w\cos(wt)dt + c$$

The integral on the RHS of the equation must be integrated by parts again,

Letting, $\dfrac{dv}{dt} = e^{\frac{t}{\tau}} \Rightarrow v = \tau e^{\frac{t}{\tau}}$ and $u = \cos(wt) \Rightarrow \dfrac{du}{dt} = -w\sin(wt)$ then,

$$\int \tau e^{\frac{t}{\tau}}\left(\frac{k_p A}{\tau}\right)w\cos(wt)dt = k_p Aw\int e^{\frac{t}{\tau}}\cos(wt)dt = k_p Aw\left[\cos(wt)\tau e^{\frac{t}{\tau}} + \int \tau e^{\frac{t}{\tau}}w\sin(wt)dt\right]$$

Combining terms,

$$e^{\frac{t}{\tau}}y = \int e^{\frac{t}{\tau}}\frac{k_p}{\tau}A\sin(wt)dt = \left(\frac{k_p A}{\tau}\right)\sin(wt)\tau e^{\frac{t}{\tau}} - k_p Aw\left[\cos(wt)\tau e^{\frac{t}{\tau}} + \int \tau e^{\frac{t}{\tau}}w\sin(wt)dt\right] + c$$

and re-arranging,

$$\tau e^{\frac{t}{\tau}}y = \int e^{\frac{t}{\tau}}k_p A\sin(wt)dt = k_p A\sin(wt)\tau e^{\frac{t}{\tau}} - k_p Aw\tau^2\left[\cos(wt)e^{\frac{t}{\tau}} + \int e^{\frac{t}{\tau}}w\sin(wt)dt\right] + c\tau$$

$$\tau e^{\frac{t}{\tau}}y = (1+w^2\tau^2)\int e^{\frac{t}{\tau}}k_p A\sin(wt)dt = k_p A\sin(wt)\tau e^{\frac{t}{\tau}} - k_p Aw\tau^2\cos(wt)e^{\frac{t}{\tau}} + c\tau$$

Therefore,

$$y = \frac{k_p A(\sin(wt) - w\tau\cos(wt))}{(1+w^2\tau^2)} + \frac{ce^{\frac{-t}{\tau}}}{(1+w^2\tau^2)}$$

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[3] $\int u\dfrac{dv}{dx}dx = uv - \int v\dfrac{du}{dx}dx$

Note:

$x\sin\alpha + y\cos\alpha = z\sin(\alpha+\phi)$ where $\tan\phi = y/x$

and $z^2 = x^2 + y^2$

Therefore,

$$y = \frac{k_p A\sqrt{1+w^2\tau^2}\,\sin(wt+\phi)}{(1+w^2\tau^2)} + \frac{c' e^{\frac{-t}{\tau}}}{(1+w^2\tau^2)}$$

The boundary conditions are, t = 0, y = 0, therefore,

$$k_p A\sqrt{1+w^2\tau^2}\,\sin(0+\phi) = k_p A(\sin(wt) - w\tau\cos(wt) = -k_p Aw\tau = -c'$$

Substituting for the boundary condition gives,

$$y = \frac{k_p A}{1+w^2\tau^2}\left[\sqrt{1+w^2\tau^2}\,\sin(wt+\phi) + w\tau e^{\frac{-t}{\tau}}\right]$$